

# Long time behavior of subcritical SQG equations in scale-invariant Sobolev spaces

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**ABSTRACT.** We consider the subcritical SQG equation in the natural scale invariant Sobolev space and prove the existence of a global attractor of optimal regularity. The proof is based on a new energy estimate in Sobolev spaces to bootstrap the regularity to the optimal level, derived by means of nonlinear lower bounds on the fractional laplacian. This estimate appears to be new in the literature, and allows a sharp use of the subcritical nature of the  $L^\infty$  bounds for this problem. As a byproduct, we obtain attractors for weak solutions as well.

## 1. Introduction

The dissipative surface quasi-geostrophic equation (SQG) describes the evolution of the potential temperature  $\theta$  on the two-dimensional horizontal boundaries of general three-dimensional quasi-geostrophic equations [6, 28]. Due to its similarities with the three-dimensional Euler and Navier-Stokes equations, it has attracted the attention of many mathematicians over the last two decades. Formulated on the two-dimensional torus  $\mathbb{T}^2 = [0, 1]^2$ , the Cauchy problem reads

$$\begin{cases} \partial_t \theta + \mathbf{u} \cdot \nabla \theta + \Lambda^\gamma \theta = f, \\ \mathbf{u} = \mathcal{R}^\perp \theta = \nabla^\perp \Lambda^{-1} \theta, \\ \theta(0) = \theta_0, \quad \int_{\mathbb{T}^2} \theta_0(x) = 0. \end{cases} \quad (\text{SQG}_\gamma)$$

Here,  $\Lambda = \sqrt{-\Delta}$  is the Zygmund operator,  $\gamma \in (0, 2)$  is a parameter measuring the strength of the diffusion, for which the diffusivity parameter has been normalized to 1, and  $f$  is a time-independent, mean-free forcing term. In this note, we will focus on the so-called *subcritical* case, when  $\gamma \in (1, 2)$ , and prove the following result.

**THEOREM 1.1.** *Let  $\gamma \in (1, 2)$  be fixed, and assume that  $f \in L^\infty \cap H^{2-\gamma}$ . The dynamical system  $S_\gamma(t)$  generated by  $(\text{SQG}_\gamma)$  on  $H^{2-\gamma}$  possesses a unique invariant global attractor  $A_\gamma$ , bounded in  $H^{2-\gamma/2}$ , and therefore compact in  $H^{2-\gamma}$ . In particular,*

$$\lim_{t \rightarrow \infty} \text{dist}_{H^{2-\gamma}}(S_\gamma(t)B, A_\gamma) = 0,$$

for every bounded set  $B \subset H^{2-\gamma}$ .

The analysis can actually be extended to weak solutions to show that the basin of attraction of  $A_\gamma$  is the whole space  $L^2$ , modulo working with multivalued dynamical systems, due to the possible non-uniqueness of weak solutions. Notice also that the assumptions on  $f$  can be relaxed.

**THEOREM 1.2.** *Let  $\gamma \in (1, 2)$  be fixed, and assume that  $f \in L^\infty \cap H^{\gamma/2}$ . The multivalued dynamical system  $S_\gamma(t)$  generated by  $(\text{SQG}_\gamma)$  on  $L^2$  possesses a unique invariant global attractor  $A_\gamma$ , bounded in*

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$H^{\gamma/2}$ , and therefore compact in  $L^2$ . In particular,

$$\lim_{t \rightarrow \infty} \text{dist}_{L^2}(S_\gamma(t)B, A_\gamma) = 0, \quad (1.1)$$

for every bounded set  $B \subset L^2$ . Furthermore, if  $f \in H^{2-\gamma}$ ,  $A_\gamma$  coincides with that of Theorem 1.1.

In the statements above,  $\text{dist}_X$  stands for the Hausdorff semidistance in  $X$  between sets, given by

$$\text{dist}_X(B, C) = \sup_{b \in B} \inf_{c \in C} \|b - c\|_X, \quad B, C \subset X.$$

The asymptotic behavior of solutions to (SQG $_\gamma$ ) in terms of attractors has been investigated by several authors in recent times. In the subcritical case  $\gamma \in (1, 2)$ , the existence of a weak global attractor in  $L^2$  was proved in [1], that is, the existence of a weakly compact, weakly attracting set for which (1.1) is replaced by the distance induced by the weak  $L^2$ -metric on bounded sets. A strong (and smooth) attractor was later constructed in [21], where the semigroup  $S_\gamma(t)$  was considered on  $H^s$ , with  $s > 2 - \gamma$ , a space above the critical regularity level. (see [21, Theorem 5.1]). The main obstructions with working in the larger space  $H^{2-\gamma}$  can be summarized as follows.

◊ Scaling invariance: if  $\theta(x, t)$  is a solution to (SQG $_\gamma$ ) with datum  $\theta_0(x)$ , then  $\theta_\lambda(x, t) = \lambda^{\gamma-1}\theta(\lambda x, \lambda^\gamma t)$  is a solution of (SQG $_\gamma$ ) with initial datum  $\theta_{0,\lambda}(x) = \lambda^{\gamma-1}\theta_0(\lambda x)$ . Therefore  $H^{2-\gamma}$  is scale-invariant, and thus the time of local existence of a solution arising from an initial datum  $\theta_0 \in H^{2-\gamma}$  is not known to depend solely on  $\|\theta_0\|_{H^{2-\gamma}}$ , and a uniform regularization with respect to initial data cannot be obtained only by exploiting short-time parabolic regularization.

◊ Maximum principles: while smooth solutions to (SQG $_\gamma$ ) automatically satisfy an a priori  $L^\infty$  bound, our case necessitates a uniform (w.r.t to initial data) regularization from  $L^2$  to  $L^\infty$ , reminiscent of De Giorgi type iterations [2–4, 31], to obtain an  $L^\infty$  absorbing set (cf. Theorem 2.1, proven in [4]).

◊ Sobolev estimates: the proofs of Theorem 1.1-1.2 rely on the existence of regular absorbing sets (i.e. bounded in higher order Sobolev spaces) for the dynamics of (SQG $_\gamma$ ). However, the scaling invariance of  $H^{2-\gamma}$  does not allow the use of the commutator estimates used in [21] (see Section 3), and a new approach based on pointwise lower bounds on the fractional laplacian [8, 9] is required (cf. Theorem 3.1). Specifically, the subcritical nature of the  $L^\infty$  control of Theorem 2.1 is used in a sharp way.

It is worth mentioning that similar results hold for the critical ( $\gamma = 1$ ) SQG equation [3, 5, 8, 15]. This (harder) case presents an additional difficulty, namely,  $L^\infty$  is also scale-invariant and a control in Hölder spaces is necessary to infer the existence of an absorbing set in  $H^1$ . While the same arguments apply here, we develop a much simpler strategy avoiding the use of Hölder spaces.

**Organization of the paper.** In Section 2 we introduce the proper functional setting and state a result on the existence of an  $L^\infty$  absorbing set, proved in [4]. We then derive a new Sobolev estimate in Section 3, based on pointwise estimates on the evolution of finite differences, and prove the existence of a bounded absorbing set in  $H^{2-\gamma}$ . The proofs of Theorems 1.1-1.2 are carried out in Section 4. We conclude with a few remarks and possible future projects in Section 5.

## 2. The subcritical SQG equation as a dynamical system

Let  $\gamma \in (1, 2)$  and assume that  $f \in L^\infty \cap H^{2-\gamma}$ . It follows from several works [10, 16, 22, 29] that for all initial data  $\theta_0 \in H^{2-\gamma}$  the initial value problem (SQG $_\gamma$ ) admits a unique global solution

$$\theta^\gamma \in C([0, \infty); H^{2-\gamma}) \cap L^2_{loc}(0, \infty; H^{2-\gamma/2}).$$

In other words, the solution operators

$$S_\gamma(t) : H^{2-\gamma} \rightarrow H^{2-\gamma}, \quad t \geq 0,$$

acting as

$$\theta_0 \mapsto S_\gamma(t)\theta_0 = \theta^\gamma(t), \quad \forall t \geq 0,$$

are well-defined and, being the forcing term autonomous, they form a semigroup of operators. By standard arguments, it is not hard to see that  $\theta^\gamma$  satisfies the energy inequality

$$\|\theta^\gamma(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\gamma/2} \theta^\gamma(s)\|_{L^2}^2 ds \leq \|\theta_0\|_{L^2}^2 + \frac{1}{\kappa_\gamma} \|f\|_{L^2}^2 t, \quad \forall t \geq 0. \quad (2.1)$$

and the decay estimate

$$\|\theta^\gamma(t)\|_{L^2} \leq \|\theta_0\|_{L^2} e^{-\kappa_\gamma t} + \frac{1}{\kappa_\gamma} \|f\|_{L^2}, \quad \forall t \geq 0, \quad (2.2)$$

where  $\kappa_\gamma \geq 1$  is a universal constant. If furthermore  $\theta_0 \in L^\infty$ , then cf. [8, 12] we have

$$\|\theta^\gamma(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} e^{-\kappa_\gamma t} + \frac{1}{\kappa_\gamma} \|f\|_{L^\infty}, \quad \forall t \geq 0. \quad (2.3)$$

Since we consider mean-zero solutions to  $(\text{SQG}_\gamma)$ , by the symbol  $H^s$  we indicate the the *homogeneous* Sobolev space of order  $s \in \mathbb{R}$ , with norm  $\|\cdot\|_{H^s} = \|\Lambda^s \cdot\|_{L^2}$ . As for the fractional laplacian, we will mainly use its representation as the singular integral

$$\Lambda^\gamma \theta(x) = c_\gamma \sum_{k \in \mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{\theta(x) - \theta(x+y)}{|y - 2\pi k|^{2+\gamma}} dy = c_\gamma \text{P.V.} \int_{\mathbb{R}^2} \frac{\theta(x) - \theta(x+y)}{|y|^{2+\gamma}} dy,$$

abusing notation and denoting by  $\theta$  the periodic extension of  $\theta$  to the whole space. The velocity vector field  $\mathbf{u}$  in  $(\text{SQG}_\gamma)$  is divergence-free and determined by  $\theta$  through the relation

$$\mathbf{u} = \mathcal{R}^\perp \theta = \nabla^\perp \Lambda^{-1} \theta = (-\partial_{x_2} \Lambda^{-1} \theta, \partial_{x_1} \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta),$$

where

$$\begin{aligned} \mathcal{R}_j \theta(x) &= \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{T}^2} \frac{y_j}{|y|^3} \theta(x+y) dy + \sum_{k \in \mathbb{Z}_*^2} \int_{\mathbb{T}^2} \left( \frac{y_j + 2\pi k_j}{|y + 2\pi k|^3} - \frac{2\pi k_j}{|2\pi k|^3} \right) \theta(x+y) dy \\ &= \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \frac{y_j}{|y|^3} \theta(x+y) dy. \end{aligned}$$

In the last line the principal value is taken both as  $|y| \rightarrow 0$  and  $|y| \rightarrow \infty$ .

**2.1.  $L^\infty$  absorbing sets.** Recall that a set  $B_0$  is *absorbing* if for every bounded set  $B \subset H^{2-\gamma}$  there exists  $t_B > 0$  such that

$$S_\gamma(t)B \subset B_0, \quad \forall t \geq t_B.$$

The following theorem was proved in [4].

**THEOREM 2.1.** *Let*

$$R_\infty = \frac{2}{\kappa_\gamma} \|f\|_{L^\infty}.$$

*The set*

$$B_\infty^\gamma = \{\varphi \in L^\infty \cap H^{2-\gamma} : \|\varphi\|_{L^\infty} \leq R_\infty\}$$

*is an absorbing set for  $S_\gamma(t)$ . Moreover,*

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_\infty^\gamma} \|S_\gamma(t) \theta_0\|_{L^\infty} \leq 2R_\infty. \quad (2.4)$$

The idea of the proof relies on the dissipative estimate (2.2) and an appropriate De Giorgi type iteration scheme, and it is carried out in details in [4, Lemma 4.2] (see also [2, 3]). In particular, it is crucial that the  $L^\infty$  norm of the solution at any positive time is controlled by the  $L^2$  norm of the initial datum and the forcing.

**REMARK 2.2.** The assumptions on  $f$  can in fact be relaxed to  $f \in L^2$  at this stage, at the cost of introducing a dependence on  $\gamma$  in the expression of  $R_\infty$  above. In this way, the radius of the  $L^\infty$  absorbing set would diverge as  $\gamma \rightarrow 1^+$ .

### 3. Sobolev estimates via nonlinear lower bounds

The main goal of this section is to establish a proper dissipative estimate of  $S_\gamma(t)$  in the phase space norm  $\|\cdot\|_{H^{2-\gamma}}$ . This is not at all trivial. Indeed, testing equation (SQG $_\gamma$ ) with  $\theta$  in  $H^{2-\gamma}$  and using commutator estimates (see e.g. [20]), we arrive at a differential inequality of the form

$$\frac{d}{dt}\|\theta\|_{H^{2-\gamma}}^2 + \|\theta\|_{H^{2-\gamma/2}}^2 \leq c\|\theta\|_{H^{2-\gamma}}\|\theta\|_{H^{2-\gamma/2}}^2 + c\|f\|_{H^{2-\gamma}}^2, \quad (3.1)$$

which does not yield any proper dissipative estimate for  $\|\theta\|_{H^{2-\gamma}}$ . To overcome this difficulty, we first proceed by pointwise estimates in the spirit of [5, 8, 9, 14], in order to be able to exploit the available nonlinear lower bounds for fractional diffusion operators. The main result of this section can be phrased as follows.

**THEOREM 3.1.** *Let  $\theta_0 \in L^\infty$ ,  $f \in L^\infty \cap H^\alpha$ ,  $\gamma \in (1, 2)$  and  $\alpha \in (0, 1)$ . Then the differential inequality*

$$\frac{d}{dt}\|\theta\|_{H^\alpha}^2 + \frac{1}{4}\|\theta\|_{H^{\alpha+\gamma/2}}^2 \leq c\left[\|\theta_0\|_{L^\infty} + \frac{1}{\kappa_\gamma}\|f\|_{L^\infty}\right]^{\frac{4\gamma}{\gamma-1}} + c\|f\|_{H^\alpha}^2 \quad (3.2)$$

*holds for every  $t > 0$ , with  $c > 0$  independent of  $\gamma$ .*

In the case  $\alpha = 2 - \gamma$ , the improvement of (3.2) with respect to (3.1) is dramatic, since we now have

$$\frac{d}{dt}\|\theta\|_{H^{2-\gamma}}^2 + \frac{1}{4}\|\theta\|_{H^{2-\gamma/2}}^2 \leq c\left[\|\theta_0\|_{L^\infty} + \frac{1}{\kappa_\gamma}\|f\|_{L^\infty}\right]^{\frac{4\gamma}{\gamma-1}} + c\|f\|_{H^{2-\gamma}}^2. \quad (3.3)$$

The above estimate makes the scale-invariant space  $H^{2-\gamma}$  treatable. Before proceeding to the proof, postponed in Section 3.2, we discuss in the next section an important consequence of the above inequality.

**3.1. Absorbing sets in scale-invariant spaces.** From estimate (3.3) and the standard Gronwall lemma, we infer that

$$\|S_\gamma(t)\theta_0\|_{H^{2-\gamma}}^2 \leq \|\theta_0\|_{H^{2-\gamma}}^2 e^{-\nu_\gamma t} + c\left[\|\theta_0\|_{L^\infty} + \frac{1}{\kappa_\gamma}\|f\|_{L^\infty}\right]^{\frac{4\gamma}{\gamma-1}} + c\|f\|_{H^{2-\gamma}}^2, \quad (3.4)$$

where  $\nu_\gamma > 0$  depends on the Poincaré constant and can clearly be made independent of  $\gamma \in (1, 2)$ . In particular, due to the existence of an  $L^\infty$  absorbing set (Theorem (2.1)), the existence of an  $H^{2-\gamma}$  absorbing set follows immediately.

**THEOREM 3.2.** *The set*

$$B_1^\gamma = \{\varphi \in H^{2-\gamma} : \|\varphi\|_{H^{2-\gamma}} \leq R_{1,\gamma}\},$$

*with*

$$R_{1,\gamma}^2 = c[2R_\infty]^{\frac{4\gamma}{\gamma-1}} + c\|f\|_{H^{2-\gamma}}^2,$$

*is absorbing for  $S_\gamma(t)$ . Moreover,*

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_1^\gamma} \left[ \|S_\gamma(t)\theta_0\|_{H^{2-\gamma}}^2 + \int_t^{t+1} \|S_\gamma(\tau)\theta_0\|_{H^{2-\gamma/2}}^2 d\tau \right] \leq 2R_{1,\gamma}^2. \quad (3.5)$$

Estimate (3.5) is derived by choosing an initial datum  $\theta_0 \in B_1^\gamma$ , integrating on  $(t, t+1)$  inequality (3.3) and exploiting the bound (3.4). We discuss the optimality (rather, the non-optimality) of the radius  $R_{1,\gamma}$  in the concluding Section 5.

**REMARK 3.3.** In [8], an estimate of similar flavor was derived in the case  $\gamma = \alpha = 1$  by considering the evolution of  $\nabla\theta$  and exploiting Hölder bounds. The approach here is somewhat different, for two main reasons linked to the nonlocal nature of  $\Lambda$ : firstly, the evolution of  $\Lambda^\alpha\theta$  is not as nice, as Leibniz differentiation does not hold anymore; secondly, the pointwise nonlinear lower bounds hold for  $\nabla\theta$ , but it is not clear whether they hold for  $\Lambda^\alpha\theta$  or not. We refer to [15] for an estimate involving Hölder norms.

**3.2. A general Sobolev estimate.** For convenience, in the course of this section we will set

$$K_\infty = \|\theta_0\|_{L^\infty} + \frac{1}{\kappa_\gamma} \|f\|_{L^\infty}, \quad (3.6)$$

so that in view of (2.3) the solution originating from  $\theta_0$  satisfies the global bound

$$\|\theta(t)\|_{L^\infty} \leq K_\infty, \quad \forall t \geq 0. \quad (3.7)$$

Consider the finite difference

$$\delta_h \theta(x, t) = \theta(x + h, t) - \theta(x, t),$$

which is periodic in both  $x$  and  $h$ , where  $x, h \in \mathbb{T}^2$ . As in [8, 14], it follows that

$$L(\delta_h \theta)^2 + D_\gamma[\delta_h \theta] = \delta_h f, \quad (3.8)$$

where  $L$  denotes the differential operator

$$L = \partial_t + \mathbf{u} \cdot \nabla_x + (\delta_h \mathbf{u}) \cdot \nabla_h + \Lambda^\gamma$$

and

$$D_\gamma[\psi](x) = c_\gamma \int_{\mathbb{R}^2} \frac{[\psi(x) - \psi(x + y)]^2}{|y|^{2+\gamma}} dy.$$

For an arbitrary  $\alpha \in (0, 1)$ , we study the evolution of the quantity  $v(x, t; h)$  defined by

$$v(x, t; h) = \frac{\delta_h \theta(x, t)}{|h|^{1+\alpha}}.$$

Notice that

$$\|\theta(t)\|_{H^\alpha}^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [v(x, t; h)]^2 dh dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[\theta(x + h, t) - \theta(x, t)]^2}{|h|^{2+2\alpha}} dh dx.$$

From (3.8) and a calculation analogous to that in [8, 14], we arrive at

$$Lv^2 + \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} = -4(1 + \alpha) \frac{h}{|h|^2} \cdot \delta_h \mathbf{u} v^2 + \frac{(\delta_h f)(\delta_h \theta)}{|h|^{2+2\alpha}}, \quad (3.9)$$

with  $\delta_h \mathbf{u} = \mathcal{R}^\perp \delta_h \theta$ . We now estimate the dissipative term  $D_\gamma[\delta_h \theta]$  from below and the nonlinear term  $\delta_h \mathbf{u}$  from above.

LEMMA 3.4. *There exists a positive constant  $\tilde{c}_\gamma$  such that*

$$D_\gamma[\delta_h \theta](x, t) \geq \tilde{c}_\gamma \frac{|\delta_h \theta(x, t)|^{2+\gamma}}{|h|^\gamma \|\theta(t)\|_{L^\infty}^\gamma}$$

*holds for any  $x, h \in \mathbb{T}^2$  and any  $t \geq 0$ .*

PROOF. For the sake of brevity, we omit the time dependence of every function below. As proven in [8, 9, 14], for  $r \geq 4|h|$  there holds

$$D_\gamma[\delta_h \theta](x) \geq \frac{c_\gamma}{r^\gamma} |\delta_h \theta(x)|^2 - cc_\gamma |\delta_h \theta(x)| \|\theta\|_{L^\infty} \frac{|h|}{r^{1+\gamma}}, \quad (3.10)$$

where  $c \geq 1$  is an absolute constant. We choose  $r > 0$  such that

$$\frac{c_\gamma}{r^\gamma} |\delta_h \theta(x)|^2 = 8cc_\gamma |\delta_h \theta(x)| \|\theta\|_{L^\infty} \frac{|h|}{r^{1+\gamma}},$$

namely,

$$r = \frac{8c \|\theta\|_{L^\infty} |h|}{|\delta_h \theta(x)|}.$$

Notice that since  $|\delta_h \theta(x)| \leq 2\|\theta\|_{L^\infty}$ , we immediately obtain that  $r \geq 4|h|$ . The result follows by plugging  $r$  back into (3.10).  $\square$

LEMMA 3.5. *Let  $r \geq 4|h|$  be arbitrarily fixed. Then*

$$|\delta_h \mathbf{u}(x, t)| \leq c \left[ r^{\gamma/2} (D_\gamma[\delta_h \theta](x, t))^{1/2} + \frac{|h| \|\theta(t)\|_{L^\infty}}{r} \right],$$

*holds pointwise in  $x, h \in \mathbb{T}^2$  and  $t \geq 0$ .*

PROOF. Let us fix  $r \geq 4|h|$ , and let  $\chi$  be a smooth radially non-increasing cutoff function that vanishes on  $|x| \leq 1$  and is identically 1 for  $|x| \geq 2$  and such that  $|\chi'| \leq 2$ . We split the vector  $\delta_h \mathbf{u}$  in an inner and an outer part

$$\delta_h \mathbf{u}(x) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} [\delta_h \theta(x+y) - \delta_h \theta(x)] dy = \delta_h \mathbf{u}_{in}(x) + \delta_h \mathbf{u}_{out}(x),$$

by using that the kernel of  $\mathcal{R}^\perp$  has zero average on the unit sphere, where

$$\delta_h \mathbf{u}_{in}(x) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} [1 - \chi(|y|/r)] [\delta_h \theta(x+y) - \delta_h \theta(x)] dy,$$

and

$$\begin{aligned} \delta_h \mathbf{u}_{out}(x) &= \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi(|y|/r) [\delta_h \theta(x+y)] dy \\ &= \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{R}^2} \delta_{-h} \left[ \frac{y^\perp}{|y|^3} \chi(|y|/r) \right] [\theta(x+y)] dy. \end{aligned}$$

For the inner piece, in light of the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} |\delta_h \mathbf{u}_{in}(x)| &\leq \frac{1}{2\pi} \int_{|y| \leq r} \frac{1}{|y|^2} |\delta_h \theta(x+y) - \delta_h \theta(x)| dy \\ &\leq \frac{1}{2\pi} \left[ \int_{|y| \leq r} \frac{1}{|y|^{2-\gamma}} \right]^{1/2} \left[ \int_{\mathbb{R}^2} \frac{(\delta_h \theta(x+y) - \delta_h \theta(x))^2}{|y|^{2+\gamma}} dy \right]^{1/2} \\ &\leq c r^{\gamma/2} (D_\gamma[\delta_h \theta](x))^{1/2}. \end{aligned} \tag{3.11}$$

Regarding the outer part, the mean value theorem entails

$$|\delta_h \mathbf{u}_{out}(x)| \leq c|h| \int_{|y| \geq r/2} \frac{|\theta(x+y)|}{|y|^3} dy \leq c \frac{|h| \|\theta\|_{L^\infty}}{r}. \tag{3.12}$$

The conclusion follows by combining (3.11) and (3.12).  $\square$

We are now ready to complete the proof of the estimate (3.2).

PROOF OF THEOREM 3.1. Without loss of generality, we may assume  $K_\infty \geq 1$ . Combining (3.6) and (3.9) with the results of the above two lemmas, we obtain the inequality

$$Lv^2 + \frac{1}{2} \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} + \tilde{c}_\gamma \frac{|\delta_h \theta|^{2+\gamma}}{|h|^{2+2\alpha+\gamma} K_\infty^\gamma} \leq c \left[ r^{\gamma/2} (D_\gamma[\delta_h \theta])^{1/2} + \frac{|h| K_\infty}{r} \right] \frac{v^2}{|h|} + \frac{(\delta_h f)(\delta_h \theta)}{|h|^{2+2\alpha}}. \tag{3.13}$$

By the Cauchy-Schwartz inequality,

$$c \left[ r^{\gamma/2} (D_\gamma[\delta_h \theta])^{1/2} + \frac{|h| K_\infty}{r} \right] \frac{v^2}{|h|} \leq \frac{1}{4} \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} + c \left[ |h|^{2\alpha} v^4 r^\gamma + \frac{K_\infty}{r} v^2 \right].$$

We now choose  $r > 0$  as

$$r = 4 \left[ \frac{4K_\infty^2}{|h|^{2\alpha} v^2} \right]^{\frac{1}{1+\gamma}},$$

so that, in particular by (3.7) we obtain

$$r = 4 \left[ \frac{4K_\infty^2}{|\delta_h \theta|^2} \right]^{\frac{1}{1+\gamma}} |h|^{\frac{2}{1+\gamma}} \geq 4|h|^{\frac{2}{1+\gamma}} \geq 4|h|,$$

since  $|h| \leq 1$  and  $\gamma > 1$ . In this way, since we assumed  $K_\infty \geq 1$ ,

$$|h|^{2\alpha} v^4 r^\gamma + \frac{K_\infty}{r} v^2 \leq 2|h|^{2\alpha} v^4 r^\gamma \leq cK_\infty^{\frac{2\gamma}{1+\gamma}} |h|^{\frac{2\alpha}{1+\gamma}} v^{2+\frac{2}{1+\gamma}},$$

and (3.13) becomes

$$Lv^2 + \frac{1}{4} \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} + \tilde{c}_\gamma \frac{|\delta_h \theta|^{2+\gamma}}{|h|^{2+2\alpha+\gamma} K_\infty^\gamma} \leq cK_\infty^{\frac{2\gamma}{1+\gamma}} |h|^{\frac{2\alpha}{1+\gamma}} v^{2+\frac{2}{1+\gamma}} + \frac{(\delta_h f)(\delta_h \theta)}{|h|^{2+2\alpha}}. \quad (3.14)$$

Using Young inequality with

$$p = \frac{1+\gamma}{2}, \quad q = \frac{1+\gamma}{\gamma-1},$$

we infer that

$$cK_\infty^{\frac{2\gamma}{1+\gamma}} |h|^{\frac{2\alpha}{1+\gamma}} v^{2+\frac{2}{1+\gamma}} \leq \tilde{c}_\gamma \frac{|\delta_h \theta|^{2+\gamma}}{|h|^{2+2\alpha+\gamma} K_\infty^\gamma} + c \frac{K_\infty^{\frac{4\gamma}{\gamma-1}}}{|h|^{2\alpha}}$$

Therefore, from (3.14) we deduce that

$$Lv^2 + \frac{1}{4} \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} \leq c \frac{K_\infty^{\frac{4\gamma}{\gamma-1}}}{|h|^{2\alpha}} + \frac{(\delta_h f)(\delta_h \theta)}{|h|^{2+2\alpha}}.$$

We integrate the above inequality first in  $h \in \mathbb{T}^2$  (which is allowed, since  $\alpha < 1$ ) and then  $x \in \mathbb{T}^2$ . Using that

$$\frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{D_\gamma[\delta_h \theta]}{|h|^{2+2\alpha}} dh dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\delta_h \Lambda^{\gamma/2} \theta|^2}{|h|^{2+2\alpha}} dh dx = \|\theta\|_{H^{\alpha+\gamma/2}}^2$$

and the estimate, valid for  $\alpha \in (0, 1)$ ,

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\delta_h f)(\delta_h \theta)}{|h|^{2+2\alpha}} dh dx &\leq \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\delta_h f|^2}{|h|^{2+2\alpha}} dh dx \right]^{1/2} \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\delta_h \theta|^2}{|h|^{2+2\alpha}} dh dx \right]^{1/2} \\ &\leq \|f\|_{H^\alpha} \|\theta\|_{H^\alpha} \leq \frac{1}{4} \|\theta\|_{H^{\alpha+\gamma/2}}^2 + c\|f\|_{H^\alpha}^2, \end{aligned}$$

we arrive at

$$\frac{d}{dt} \|\theta\|_{H^\alpha}^2 + \frac{1}{4} \|\theta\|_{H^{\alpha+\gamma/2}}^2 \leq cK_\infty^{\frac{4\gamma}{\gamma-1}} + c\|f\|_{H^\alpha}^2.$$

This is precisely (3.2), and the proof is concluded.  $\square$

#### 4. The global attractor

A sufficient condition for the existence of the global attractor (the unique compact set of the phase space that is invariant and attracting) for a dynamical system is the existence of a compact absorbing set [19, 30, 32]. Moreover, being the attractor the minimal set in the class of closed attracting sets, it is contained in any (closed) absorbing set. In particular, the attractor inherits the regularity property of the absorbing set, namely, the existence of regular (i.e. bounded in higher order Sobolev spaces) absorbing sets translate into the existence of a regular attractor. We prove Theorem 1.1 in the next Section 4.1, and Theorem 1.2 in the subsequent Section 4.2, by using again the estimate (3.2) several times.

In the course of this section, we will often make use of the fractional product inequality [23]

$$\|\Lambda^s(\varphi\psi)\|_{L^p} \leq c[\|\varphi\|_{L^{p_1}} \|\Lambda^s \psi\|_{L^{p_2}} + \|\Lambda^s \varphi\|_{L^{p_3}} \|\psi\|_{L^{p_4}}], \quad (4.1)$$

valid for  $s > 0$ ,  $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$  and  $p_2, p_3 \in (1, \infty)$ , the commutator estimate [24]

$$\|\Lambda^s(\varphi\psi) - \varphi\Lambda^s \psi\|_{L^p} \leq c[\|\nabla \varphi\|_{L^{p_1}} \|\Lambda^{s-1} \psi\|_{L^{p_2}} + \|\Lambda^s \varphi\|_{L^{p_3}} \|\psi\|_{L^{p_4}}], \quad (4.2)$$

with the same constraints as above, and the Sobolev embedding

$$\|\varphi\|_{L^p} \leq c\|\Lambda^{1-2/p}\varphi\|_{L^2}, \quad (4.3)$$

with  $p \in [2, \infty)$ .

**4.1. Regular absorbing sets.** The existence and regularity of the attractor in Theorem 1.1 follow from the existence of an absorbing set bounded in  $H^{2-\gamma/2}$ .

THEOREM 4.1. *The set*

$$B_2^\gamma = \left\{ \varphi \in H^{2-\gamma/2} : \|\varphi\|_{H^{2-\gamma/2}} \leq R_{2,\gamma} \right\}$$

with

$$R_{2,\gamma}^2 = c \left[ 2R_{1,\gamma}^2 + \|f\|_{H^{2-\gamma}}^2 \right] e^{cR_{1,\gamma}^2},$$

is absorbing for  $S_\gamma(t)$ . Moreover,

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_2^\gamma} \|S_\gamma(t)\theta_0\|_{H^{2-\gamma/2}}^2 \leq \mathcal{Q}(R_{2,\gamma}), \quad (4.4)$$

where  $\mathcal{Q}(\cdot)$  is a positive increasing function with  $\mathcal{Q}(0) = 0$ .

PROOF. Clearly, it is enough to show that  $B_2^\gamma$  absorbs  $B_1^\gamma$ , the  $H^{2-\gamma}$  absorbing set obtained in Theorem 3.2. If  $\theta_0 \in B_1^\gamma$ , then (3.5) implies that

$$\sup_{t \geq 0} \int_t^{t+1} \|S_\gamma(\tau)\theta_0\|_{H^{2-\gamma/2}}^2 d\tau \leq 2R_{1,\gamma}^2. \quad (4.5)$$

By testing (SQG $_\gamma$ ) with  $\theta$  in  $H^{2-\gamma/2}$  and using standard arguments, we deduce that

$$\frac{d}{dt} \|\theta\|_{H^{2-\gamma/2}}^2 + \|\theta\|_{H^2}^2 \leq \|f\|_{H^{2-\gamma}}^2 + 2 \left| \int_{\mathbb{T}^2} \left[ \Lambda^{2-\gamma/2}(\mathbf{u} \cdot \nabla \theta) - \mathbf{u} \cdot \nabla \Lambda^{2-\gamma/2} \theta \right] \Lambda^{2-\gamma/2} \theta dx \right|.$$

By means of the commutator estimate (4.2),

$$\|\Lambda^{2-\gamma/2}(\varphi\psi) - \varphi\Lambda^{2-\gamma/2}\psi\|_{L^2} \leq c \left[ \|\nabla \varphi\|_{L^{4/\gamma}} \|\Lambda^{1-\gamma/2}\psi\|_{L^{\frac{4}{2-\gamma}}} + \|\Lambda^{2-\gamma/2}\varphi\|_{L^{\frac{4}{2-\gamma}}} \|\psi\|_{L^{4/\gamma}} \right],$$

and the two-dimensional Sobolev inequality

$$\|\varphi\|_{L^p} \leq c\|\Lambda^{1-2/p}\varphi\|_{L^2}, \quad p \in [2, \infty),$$

we therefore have

$$\begin{aligned} \frac{d}{dt} \|\theta\|_{H^{2-\gamma/2}}^2 + \|\theta\|_{H^2}^2 &\leq \|f\|_{H^{2-\gamma}}^2 + c\|\theta\|_{H^{2-\gamma/2}} \|\Lambda\theta\|_{L^{4/\gamma}} \|\Lambda^{2-\gamma/2}\theta\|_{L^{\frac{4}{2-\gamma}}} \\ &\leq \|f\|_{H^{2-\gamma}}^2 + c\|\theta\|_{H^{2-\gamma/2}}^2 \|\theta\|_{H^2} \\ &\leq \|f\|_{H^{2-\gamma}}^2 + c\|\theta\|_{H^{2-\gamma/2}}^4 + \frac{1}{2}\|\theta\|_{H^2}^2. \end{aligned}$$

Hence,

$$\frac{d}{dt} \|\theta\|_{H^{2-\gamma/2}}^2 + \frac{1}{2}\|\theta\|_{H^2}^2 \leq \|f\|_{H^{2-\gamma}}^2 + c\|\theta\|_{H^{2-\gamma/2}}^4. \quad (4.6)$$

Thanks to the local integrability (4.5) and the above differential inequality, the uniform Gronwall lemma implies

$$\|S_\gamma(t)\theta_0\|_{H^{2-\gamma/2}}^2 \leq c \left[ 2R_{1,\gamma}^2 + \|f\|_{H^{2-\gamma}}^2 \right] e^{cR_{1,\gamma}^2}, \quad \forall t \geq 1. \quad (4.7)$$

Thus, setting

$$R_{2,\gamma}^2 := c \left[ 2R_{1,\gamma}^2 + \|f\|_{H^{2-\gamma}}^2 \right] e^{cR_{1,\gamma}^2},$$

we obtain that

$$S_\gamma(t)B_1^\gamma \subset B_2^\gamma, \quad \forall t \geq 1,$$



as we wanted. Concerning estimate (4.4), it is clear that it holds for  $t \geq 1$  from (4.7) and by integrating (4.6) on  $(t, t+1)$ . For  $t < 1$ , it suffices to use (4.5) and the standard Gronwall lemma on the time interval  $(0, t)$ , applied to (4.6).  $\square$

The existence of a compact absorbing set is well-known to be sufficient for the existence of the global attractor. However, due to the possible lack of continuity of the map  $S_\gamma(t) : H^{2-\gamma} \rightarrow H^{2-\gamma}$  for fixed  $t > 0$ , the invariance of  $A_\gamma$  requires some care. In fact, to conclude the proof of Theorem 1.1, it is enough to prove continuity on the regular absorbing set  $B_{2,\gamma}$ . Our next goal is then to establish the following.

**PROPOSITION 4.2.** *Let  $\gamma \in (1, 2)$ . For each fixed  $t \geq 0$ ,  $S_\gamma(t)$  is Lipschitz-continuous on  $B_2^\gamma$  in the topology of  $H^{2-\gamma}$  and*

$$\sup_{\theta_{0,i} \in B_2^\gamma} \|S_\gamma(t)\theta_{0,1} - S_\gamma(t)\theta_{0,2}\|_{H^{2-\gamma}} \leq e^{\mathcal{Q}(R_{2,\gamma})t} \|\theta_{0,1} - \theta_{0,2}\|_{H^{2-\gamma}}, \quad \forall t \geq 0, \quad (4.8)$$

where  $\mathcal{Q}(\cdot)$  is a positive increasing function with  $\mathcal{Q}(0) = 0$ .

**PROOF.** Denote by  $\theta_i(t) = S_\gamma(t)\theta_{0,i}$ ,  $i = 1, 2$ , two solutions emanating from initial data  $\theta_{0,i} \in B_2^\gamma$ . Their difference  $\bar{\theta} = \theta_1 - \theta_2$  solves the equation

$$\partial_t \bar{\theta} + \mathbf{u}_1 \cdot \nabla \bar{\theta} + \bar{\mathbf{u}} \cdot \nabla \theta_2 + \Lambda^\gamma \bar{\theta} = 0.$$

Testing the above equation with  $\bar{\theta}$  in  $H^{2-\gamma}$  yields

$$\frac{1}{2} \frac{d}{dt} \|\bar{\theta}\|_{H^{2-\gamma}}^2 + \|\bar{\theta}\|_{H^{2-\gamma/2}}^2 \leq \left| \int_{\mathbb{T}^2} \Lambda^{2-\gamma} (\mathbf{u}_1 \cdot \nabla \bar{\theta}) \Lambda^{2-\gamma} \bar{\theta} dx \right| + \left| \int_{\mathbb{T}^2} \Lambda^{2-\gamma} (\bar{\mathbf{u}} \cdot \nabla \theta_2) \Lambda^{2-\gamma} \bar{\theta} dx \right|. \quad (4.9)$$

We estimate the two terms in right-hand-side above separately. Concerning the first one, we use (4.2) and (4.3) to get

$$\begin{aligned} \left| \int_{\mathbb{T}^2} \Lambda^{2-\gamma} (\mathbf{u}_1 \cdot \nabla \bar{\theta}) \Lambda^{2-\gamma} \bar{\theta} dx \right| &= \left| \int_{\mathbb{T}^2} [\Lambda^{2-\gamma} (\mathbf{u}_1 \cdot \nabla \bar{\theta}) - \mathbf{u}_1 \cdot \Lambda^{2-\gamma} \nabla \bar{\theta}] \Lambda^{2-\gamma} \bar{\theta} dx \right| \\ &\leq \|\Lambda^{2-\gamma} (\mathbf{u}_1 \cdot \nabla \bar{\theta}) - \mathbf{u}_1 \cdot \Lambda^{2-\gamma} \nabla \bar{\theta}\|_{L^{\frac{4}{2-\gamma}}} \|\Lambda^{2-\gamma} \bar{\theta}\|_{L^{\frac{4}{2-\gamma}}} \\ &\leq c \left[ \|\nabla \mathbf{u}_1\|_{L^{4/\gamma}} \|\Lambda^{2-\gamma} \bar{\theta}\|_{L^2} + \|\Lambda^{2-\gamma} \mathbf{u}_1\|_{L^{\frac{4}{2-\gamma}}} \|\nabla \bar{\theta}\|_{L^{2/\gamma}} \right] \|\bar{\theta}\|_{H^{2-\gamma/2}} \\ &\leq c \|\theta_1\|_{H^{2-\gamma/2}} \|\bar{\theta}\|_{H^{2-\gamma}} \|\bar{\theta}\|_{H^{2-\gamma/2}}, \end{aligned}$$

while for the second we exploit (4.1) and (4.3) to obtain

$$\begin{aligned} \left| \int_{\mathbb{T}^2} \Lambda^{2-\gamma} (\bar{\mathbf{u}} \cdot \nabla \theta_2) \Lambda^{2-\gamma} \bar{\theta} dx \right| &= \left| \int_{\mathbb{T}^2} \Lambda^{2-3\gamma/2} (\bar{\mathbf{u}} \cdot \nabla \theta_2) \Lambda^{2-\gamma/2} \bar{\theta} dx \right| \\ &\leq c \|\Lambda^{2-3\gamma/2} (\bar{\mathbf{u}} \cdot \nabla \theta_2)\|_{L^2} \|\Lambda^{2-\gamma/2} \bar{\theta}\|_{L^2} \\ &\leq c \left[ \|\bar{\mathbf{u}}\|_{L^{\frac{2}{\gamma-1}}} \|\Lambda^{3-3\gamma/2} \theta_2\|_{L^{\frac{2}{2-\gamma}}} + \|\Lambda^{2-3\gamma/2} \bar{\mathbf{u}}\|_{L^{\frac{4}{2-\gamma}}} \|\nabla \theta_2\|_{L^{4/\gamma}} \right] \|\bar{\theta}\|_{H^{2-\gamma/2}} \\ &\leq c \|\theta_2\|_{H^{2-\gamma/2}} \|\bar{\theta}\|_{H^{2-\gamma}} \|\bar{\theta}\|_{H^{2-\gamma/2}}. \end{aligned}$$

In view of the above estimates and using Young inequality, (4.9) becomes

$$\frac{d}{dt} \|\bar{\theta}\|_{H^{2-\gamma}}^2 + \|\bar{\theta}\|_{H^{2-\gamma/2}}^2 \leq c [\|\theta_1\|_{H^{2-\gamma/2}}^2 + \|\theta_2\|_{H^{2-\gamma/2}}^2] \|\bar{\theta}\|_{H^{2-\gamma}}^2.$$

In light of (4.4), the continuous dependence estimate (4.8) follows from a further application of the Gronwall inequality.  $\square$

**4.2. Global attractors for weak solutions.** A viscosity solution to  $(\text{SQG}_\gamma)$  is a mean free function  $\theta^\gamma \in C([0, \infty); L^2)$  that satisfies  $(\text{SQG}_\gamma)$  in the sense of distributions, and such that there exist sequences  $\varepsilon_n \rightarrow 0$  and  $\theta_n^\gamma$  satisfying

$$\begin{cases} \partial_t \theta_n^\gamma + \mathbf{u}_n^\gamma \cdot \nabla \theta_n^\gamma + \Lambda^\gamma \theta_n^\gamma - \varepsilon_n \Delta \theta_n^\gamma = f, \\ \mathbf{u}_n^\gamma = \mathcal{R}^\perp \theta_n^\gamma = \nabla^\perp \Lambda^{-1} \theta_n^\gamma, \end{cases}$$

such that  $\theta_n^\gamma \rightarrow \theta^\gamma$  in  $C_w([0, T]; L^2)$ , for every  $T > 0$  and  $\theta_n^\gamma(0) \rightarrow \theta(0)$  strongly in  $L^2$ . From [12], it follows that for any  $\theta_0 \in L^2$ , a (possibly non-unique) viscosity solution to  $(\text{SQG}_\gamma)$  exists. The fact that viscosity solutions are strongly continuous is a consequence of the fact that they satisfy the energy equality (see [3, 7] for a proof in the critical case). Following the approach in [13, 27], for  $t \geq 0$  and each  $\theta_0 \in L^2$  we define the set-valued maps  $S_\gamma(t) : L^2 \rightarrow 2^{L^2}$ , still denoted as the single-valued ones,

$$S_\gamma(t)\theta_0 = \{\theta^\gamma(t) : \theta^\gamma(\cdot) \text{ is a viscosity solution to } (\text{SQG}_\gamma) \text{ with } \theta^\gamma(0) = \theta_0\}.$$

Similarly to the critical case investigated in [15], it is possible to show that translations and concatenations of viscosity solutions are still viscosity solutions, so that  $S_\gamma(t)$  satisfies the semigroup property

$$S_\gamma(t + \tau) = S_\gamma(t)S_\gamma(\tau), \quad \forall t, \tau \geq 0.$$

Moreover, the graph of  $S_\gamma(t)$  is closed, namely for any  $t \geq 0$  the following implication holds true:

$$\theta_{0,n} \rightarrow \theta_0, \quad S_\gamma(t)\theta_{0,n} \ni \theta_n \rightarrow \theta \quad \Rightarrow \quad \theta \in S_\gamma(t)\theta_0.$$

Above, limits are understood in the strong topology of  $L^2$ . Therefore, to prove the existence of the global attractor is again sufficient to exhibit a compact absorbing set. To begin with, (2.2) implies the existence of an  $L^2$  bounded absorbing set

$$B_0^\gamma = \left\{ \varphi \in L^2 : \|\varphi\|_{L^2} \leq \frac{2}{\kappa_\gamma} \|f\|_{L^2} \right\}.$$

In addition, it is not hard to see from (2.1), which holds for viscosity solutions (cf. [3]), we also gain time integrability whenever  $\theta_0 \in B_0^\gamma$ , namely

$$\sup_{t \geq 0} \int_t^{t+1} \|S_\gamma(\tau)\theta_0\|_{H^{\gamma/2}}^2 d\tau \leq \frac{4}{\kappa_\gamma} \|f\|_{L^2}^2, \quad (4.10)$$

where  $\|S_\gamma(\tau)\theta_0\|_{H^{\gamma/2}}^2$  has to be understood as the supremum over all the elements in the set  $S_\gamma(\tau)\theta_0$ . Once a uniform  $L^2$  estimate is available, we can proceed as in Section 2 and deduce that the set  $B_\infty^\gamma$  in (2.4) defines an  $L^\infty$  absorbing set for the multivalued case as well. It is crucial here that the norm of  $\theta_0$  in  $L^2$  controls the  $L^\infty$  norm of the solutions for all positive times. As the next step, we can assume  $\theta_0 \in B_\infty^\gamma$  and apply (3.2) with  $\alpha = \gamma/2$ , that is

$$\frac{d}{dt} \|\theta\|_{H^{\gamma/2}}^2 + \frac{1}{4} \|\theta\|_{H^\gamma}^2 \leq c \left[ \|\theta_0\|_{L^\infty} + \frac{1}{\kappa_\gamma} \|f\|_{L^\infty} \right]^{\frac{4\gamma}{\gamma-1}} + c \|f\|_{H^{\gamma/2}}^2. \quad (4.11)$$

By neglecting the positive term  $\|\theta\|_{H^\gamma}^2$ , using (4.10) and the uniform Gronwall lemma we have that

$$\|S_\gamma(t)\theta_0\|_{H^{\gamma/2}}^2 \leq c \left[ \frac{3}{\kappa_\gamma} \|f\|_{L^\infty} \right]^{\frac{4\gamma}{\gamma-1}} + c \|f\|_{H^{\gamma/2}}^2, \quad \forall t \geq 1.$$

In other words, the set

$$B_{1/2}^\gamma = \left\{ \varphi \in H^{\gamma/2} : \|\varphi\|_{H^{\gamma/2}} \leq c [2R_\infty]^{\frac{4\gamma}{\gamma-1}} + c \|f\|_{H^{\gamma/2}}^2 \right\}$$

is absorbing for  $S_\gamma(t)$ , and in particular compact in  $L^2$ . This concludes the proof of the first part of Theorem 1.2, that is, the existence of the global attractor bounded in  $H^{\gamma/2}$ . Concerning the second part, note that (4.11) also provides time integrability of the  $H^\gamma$  norm of the solution. Since  $\gamma \in (1, 2)$ , the inclusion  $H^\gamma \subset H^{2-\gamma}$  holds, so that time integrability in  $H^{2-\gamma}$  follows from the Poincaré inequality. At this point,

(3.3) and the uniform Gronwall lemma yields the existence of an  $H^{2-\gamma}$  absorbing set of comparable size of that in Theorem 3.2, on which the restriction of  $S_\gamma(t)$  is single-valued. By arguing as in Section 4.1, the regularity of the global attractor can therefore be bootstrapped to  $H^{2-\gamma/2}$  and the proof of Theorem 1.2 is achieved.

## 5. Concluding remarks and further developments

We conclude this article with a few remarks and possible open problems.

**5.1. Optimality of the absorbing set.** The dependence of the absorbing set with respect to  $\gamma$  can certainly be improved. In our case, by exploiting only the  $L^\infty$  maximum principle, we chose a radius such that  $R_{1,\gamma} \rightarrow \infty$  as  $\gamma \rightarrow 1$ . The results of [5] indicate that, instead,  $R_{1,\gamma}$  can be made uniformly bounded for  $\gamma \in [1, 2)$ . The reason for this is fairly easy to explain: due to the scale-invariance of the  $L^\infty$  norm in the critical case, the existence of an  $H^1$  absorbing set for  $S_1(t)$  requires the existence of a  $C^\beta$  absorbing set, for some  $\beta \in (0, 1)$  small. Here, no  $C^\beta$  estimate is needed in principle, as the  $L^\infty$  norm provides a strong enough control. By adapting the techniques of [5], one could also prove the existence of an absorbing set consisting of Hölder continuous functions, thus leading to a better choice of  $R_{1,\gamma}$ , at the cost of significantly more involved estimates.

**5.2. Sobolev estimates involving Hölder norms.** A version of (3.2) involving Hölder norms also holds, and can be derived by the same arguments of Section 3. More precisely, let  $\gamma \in (0, 2)$ ,  $\alpha \in (0, 1)$  and assume that, for some

$$\beta \in (\max\{1 - \gamma, 0\}, 1)$$

we have the a priori control

$$[\theta(t)]_{C^\beta} \leq K_\beta, \quad \forall t \geq 0. \quad (5.1)$$

Then the differential inequality

$$\frac{d}{dt} \|\theta\|_{H^\alpha}^2 + \frac{1}{4} \|\theta\|_{H^{\alpha+\gamma/2}}^2 \leq c K_\beta^{\frac{4\gamma}{\gamma+\beta-1}} + \|f\|_{H^\alpha}^2 \quad (5.2)$$

holds true for every  $t \geq 0$  (for a proof, see [15]). Typically,  $K_\beta$  should be a function of  $f$  and  $\theta_0$ , similar to  $K_\infty$  in (3.6), independent of  $\gamma$ . The requirement that  $\beta > 1 - \gamma$  in the supercritical case  $\gamma \in (0, 1)$  is consistent with the regularity criteria derived in [11, 17], namely, an a priori bound of the form (5.1) leads to regularity. Unfortunately, no such bound is known for  $\gamma < 1$ . For  $\gamma = 1$  this type of bounds are known [2, 9, 18, 25, 26], and (5.2) with  $\alpha = 1/2$  has been recently used in [15]. In the case  $\gamma \in (1, 2)$ , this is precisely the second step (after what was discussed in Section 5.1) towards a choice of  $R_{1,\gamma}$ , uniformly bounded with respect to  $\gamma$ .

**5.3. Stability with respect to  $\gamma$ .** An interesting open problem is to study the behavior of the attractors  $A_\gamma$  as  $\gamma \rightarrow 1^+$ , in terms of upper semicontinuity in the  $H^1$  topology, namely to study the validity of the limit

$$\lim_{\gamma \rightarrow 1^+} \text{dist}_{H^1}(A_\gamma, A_1) = 0.$$

Notice that the attractors  $A_\gamma$  are slightly less regular than  $A_1$ , being attractors in the phase space  $H^{2-\gamma}$ , which strictly contains  $H^1$ . They are nonetheless bounded in  $H^{2-\gamma/2}$  (see Theorem 1.1) and it is essential to have bounds on  $\|A_\gamma\|_{H^{2-\gamma/2}}$  that are independent of  $\gamma$ , which would follow by carrying out the program in Sections 5.1-5.2.

**5.4. Uniform fractal dimension estimates.** A second open problem consists in investigating the finite-dimensionality of the attractors  $A_\gamma$ . It is also plausible that uniform bounds on  $A_\gamma$  would lead to uniform bounds on their fractal dimension, thus improving the results of [33]. In particular, such bounds (in the fractal dimension induced by the topology of  $H^{2-\gamma}$ ) should agree, as  $\gamma \rightarrow 1^+$ , with those derived in [8].

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